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DECOMPOSABLE ON KAEHLERIAN MANIFOLDS OF CONFORMAL RECURRENT CURVATURE TENSOR

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Abstract: Adati and Miyazawa (1967), have studied on a Riemannian space with recurrent conformally curvature and Deszcz (1976), has studied on semi-composable conformally recurrent and conformally birecurrent Riemannian spaces. After then, Negi (2017) have calculated Theorems on almost product and decomposable spaces. In this paper, we define and study decomposition on Kaehlerian manifolds of conformal recurrent curvature tensor and some theorems are established. Also, we have proved that if a Kaehlerian manifold k_n of recurrent conformal curvature is decomposable then the decomposition space Ω_{n-r} is Einstein and if a Kaehlerian conformally recurrent manifold k_n is decomposable then the recurrence vector is a gradient or the decomposition space Ω_r has constant curvature.

Keywords and Phrases: Conformal curvature, Recurrent, Riemannian space and Kaehlerian Manifold

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1. Introduction

A Riemannian space Ω_n is decomposable (Walker 1950), if it is expressed as a product $\Omega_r \times \Omega_{n-r}$ for some r, that is, if coordinates can be found so that it's metric takes the form:

$$ds^2 = \sum_{a,b=1}^{r} g_{ab} dx^a dx^b + \sum_{\lambda,\mu=r+1}^{n} g_{\lambda,\mu} dx^\lambda dx^\mu \tag{1.1}$$

where g_{ab} is function of $x^1, x^2, ..., x^r$ only, and the $g_{\lambda\mu}$ is function of is function of $x^{r+1}, x^{r+2}, ..., x^n$ only.

The equation (1.1) of two parts is the metrics of Ω_r and Ω_{n-r} called decomposable spaces of Ω_n. It is evident, from (1.1), that the Christoffel's symbols, the components of the curvature tensor, the Ricci tensor and their covariant derivatives in Ω_n are zero unless all the subscripts belong to the same suffix range 1,2,...,r or r+1,r+1,...,n. In case, all the subscripts belong to the same suffix range, say, 1,2,...,r then the symbols and tensor components are the same for Ω_r as for Ω_n and covariant differentiation in Ω_r is the same as in Ω_n with respect to $x^1, x^2, ..., x^r$.

If one of the decomposition spaces, say, Ω_{n-r} is flat then Ω_n is described as a flat extension of Ω_r. A Riemannian space $\Omega_n (n > 3)$ whose conformal curvature tensor defined by:

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}) \tag{1.2}$$

satisfies the relation:

$$C_{hijk,l} - \delta_l C_{hijk} = 0 \tag{1.3}$$

for some non-zero vector δ_l is said to be a space of recurrent conformal curvature or conformally recurrent space. If a conformally recurrent space Ω_n is decomposable into a product $\Omega_r \times \Omega_{n-r}$ then one of the decomposable spaces is flat and the other is a space of recurrent curvature or both are spaces of constant curvature.

2. Decomposable on Kaehlerian Manifolds of Conformal Recurrent Curvature Tensor

We have a space of recurrent conformal curvature on Kaehlerian manifold K_n, as decomposition spaces, that is, as a product $\Omega_r \times \Omega_{n-r}$ with its metric in the form (1.1). The recurrence vector δ_l is, without loss of simplification, understood to be non-zero for some suffix a on the range 1,2,...,r.

Theorem 2.1. *If a Kaehlerian manifold K_n of recurrent conformal curvature is decomposable then the decomposition space Ω_{n-r} is Einstein.*

Proof. Let a Riemannian space Ω_n be there of recurrent conformal curvature. Equation (1.3) with (1.2) and multiplying by g^{hl}, we have

$$\begin{aligned} &R^l_{ijk,l} - \frac{1}{n-2}(R_{ij,k} - R_{ik,j} + g_{ij}R^j_{k,l} - g_{ik}R^l_{j,i}) + \frac{1}{(n-1)(n-2)}(g_{hk}R_{,k} - g_{hj}R_{,j}) \\ &= \delta_l R^i_{ijk} - \frac{1}{n-2}(\delta_k R_{ij} - \delta_j R_{ik} + g_{ij}\delta_l R^l_K - g_{ik}\delta_j R^i_j) + \frac{R}{(n-1)(n-2)}(\delta_k g_{ij} - \delta_j g_{ik}) \end{aligned} \tag{2.1}$$

In virtue of the relations $R^l_{ijk,l} = R_{ij,k} - R_{ik,j}$ and $R^l_{j,i} = \frac{1}{2}R_{,k}$ takes the forms

$$\frac{(n-3)}{(n-2)}[(R_{ij,k} - R_{ik,j}) - \frac{1}{2(n-1)}(g_{ij}R_{,k} - g_{ik}R_{,j})] = \delta_l R^i_{ijk} - \frac{1}{n-2} \times (\delta_k Rij - \delta_j R_{ik} + g_{ij}\delta_l R^l_k - g_{ik}\delta_j R^i_j) + \frac{R}{(n-1)(n-2)}(\delta_k g_{ij} - \delta_j g_{ik}) \tag{2.2}$$

Put $i = \lambda, j = \mu$ and $\delta = a$ in (2.2) which then reduced to

$$\frac{(n-3)}{2(n-1)}g_{\lambda\mu}R_{,a} = \delta_a R_{\lambda\mu} + g_{\lambda\mu}\delta_l R^l_a - \frac{R}{(n-1)}\delta_a R_{\lambda\mu}. \tag{2.3}$$

Multiplying (2.3) by $g^{\lambda\mu}$ yields

$$\frac{(n-3)(n-r)}{2(n-1)}R_{,a} = \delta_a R^a + (n-r)\delta_l R^\lambda_\mu - \frac{(n-r)}{n-1}R\delta_a$$

Hence we find

$$\delta_l R^l_a = \frac{n-3}{2(n-1)}R_{,a} - \frac{1}{n-r}\delta_a R^a + \frac{R}{n-1}\delta_{a.} \tag{2.4}$$

It therefore follows from (2.3) and (2.4) that

$$\delta_a(R_{\lambda\mu} - \frac{R^a}{n-r}g_{\lambda\mu.}) = 0.$$

Since $\delta_a \neq 0$ for some a, we have

$$R_{\lambda\mu} = \frac{R^a}{n-r}g_{\lambda\mu.}$$

That means that the decomposition space Ω_{n-r} is Einstein. Again, we have

Theorem 2.2. *In a decomposable space $\Omega_n = \Omega_r \times \Omega_{n-r}$ of recurrent conformal curvature on Kaehlerian manifold k_r has its curvature tensor satisfying the relation: $R_{pqrs,t} = \delta_l T_{pqrs},$ where*

$$T_{pqrs} = R_{pqrs} + \frac{R^a}{(n-r)(n-r-1)}(g_{ps}g_{qr} - g_{pr}g_{qs}). \tag{2.5}$$

Proof. We have use of (1.2) and noted (1.3), then

$$R_{hijk,l} - \frac{1}{n-2}(g_{hk}R_{ij,l} - g_{hj}R_{ik,l} + g_{ij}R_{hk,l} - g_{ik}R_{hj,l}) + \frac{1}{(n-1)(n-2)}R_{,l}\,(g_{kh}g_{ij} - g_{hj}g_{ih})$$
$$= \delta_l[R_{hijk} - \frac{1}{n-2}g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - (g_{hj}g_{ik})]. \tag{2.6}$$

Put $h = p, i = \lambda, j = q, k = \mu$, and $l = r$ in (2.6). This admits

$$g_{\lambda\mu}R_{pq,r} - \frac{1}{n-1}g_{\lambda\mu}g_{pq}R_{,r} = \delta_r[g_{\lambda\mu}R_{pq} + g_{pq}R_{\lambda\mu} - \frac{R}{n-1}g_{\lambda\mu}g_{pq}].$$

On applying theorem (2.1), then above is

$$R_{pq,r} = \frac{1}{n-1}g_{pq}R_{,r} + \delta_r[R_{pq} + \frac{R^a}{n-r}g_{pq} - \frac{R}{n-1}g_{pq}]. \tag{2.7}$$

Setting all the subscripts h, i, j, k, l in (2.6) from the same suffix range $1, 2, ..., r$ and taking into account (2.7) we see that

$$\begin{aligned} &R_{pqrs,t} - \frac{1}{(n-1)(n-2)}R_{,t}(g_{ps}g_{qr} - g_{pr}g_{qs}) \\ &= \delta_l R_{pqrs} + \frac{1}{n-2}\delta_t(\frac{2R^a}{n-r} - \frac{R}{n-1})(g_{ps}g_{qr} - g_{pr}g_{qs}) \end{aligned} \tag{2.8}$$

In consequence of the relation:

$$\frac{1}{(n-1)}R_{,t} = -\frac{n-2r}{(n-r)(n-r-1)}R^a\delta_t + \frac{R}{n-1}\delta_t$$

Equation (2.8) simplifies to

$$R_{pqrs,t} = \delta_t[R_{pqrs} + \frac{R^a}{(n-r)(n-r-1)}(g_{ps}g_{qr} - g_{pr}g_{qs})] = \delta_t T_{pqrs.}$$

This completes the proof.
From the above two theorems, then we have the followings:

Corollary 2.1. *The recurrent conformal curvature on Kaehlerian manifold K_n of a Riemannian space Ω_n satisfies the identity:*

$$R_{hijk,lm} - R_{hijk,ml} + R_{jkim,hl} - R_{jkim,lh} - R_{imhl,jk} - R_{imhl,kj} = 0$$

Corollary 2.2. *If $a_{\alpha\beta}, b_\alpha$ are number satisfying:$a_{\alpha\beta} = a_{\beta\alpha}, a_{\beta\gamma}b_\alpha + a_{\gamma\alpha}b_\beta + a_{\alpha\beta}b_\gamma = 0$ for $\alpha, \beta, \gamma = 1, 2, ..., n$, then all the $a_{\alpha\beta}$ are non-zero or all the b_α are zero.*
Now, we have the following:

Theorem 2.3. *If a Kaehlerian conformally recurrent manifold K_n is decomposable, then the recurrence vector is a gradient or the decomposition space Ω_r has constant curvature.*

Proof. Since $R^*_{,P} = 0(R^*$, scalar curvature of K_{n-r} is differentiated covariantly with respect to x^{tu}, coordinate of K_r, it is obvious from theorem (2.1) that

$$R_{pqrs,tu} = \delta_{t,u} T_{pqrs} + \delta_t \delta_u T_{pqrs.} \tag{2.9}$$

where

$$\delta_{tu} = \delta_{t,u} - \delta_{u,t}. \tag{2.10}$$

There of (2.9) and Theorem 2.3 yield:

$$\delta_{tu} T_{pqrs} + \delta_{pq} T_{rstu} + \delta_{rs} T_{tupq} = 0.$$

This is of the form of Theorem 2.4, because of $T_{pqrs} = T_{rspq}$. We therefore appear at the conditions:
Either $\delta_{tu} = 0$ or $T_{pqrs} = 0$. In the first case δ_t is a gradient while in the second case, then

$$R_{pqrs} = \frac{R^*}{(n-r)(n-r-1)}(g_{pr}g_{qs} - g_{ps}g_{qr}),$$

Which implies that K_r has constant curvature.

Theorem 2.4. *If the recurrence vector of a decomposable space K_n of recurrent conformal curvature on a Kaehlerian manifolds be a gradient then K_r has constant curvature or $K_{(n-r)}$ has zero scalar curvature.*
Proof. Let the recurrence vector δ_t be a gradient. Equation (2.10) then gives $\delta_{tu} = 0$ and (2.9) become:

$$R_{pqrs,tu} - R_{pqrs,ut} = 0$$

This, with the aid of the Ricci identity, is written:

$$R_{aqrs}R^a_{ptu} + R_{pars}R^a_{qtu} + R_{pqas}R^a_{rtu} + R_{pqra}R^a_{atu} = 0. \tag{2.11}$$

Differentiating (2.11) covariantly with respect to x^* and using (2.5) and (2.11) we obtain:

$$\begin{aligned}\frac{R^{(*)}}{(n-r)(n-r-1)}\delta_u(&g_{pt}R_{uqrs} - g_{p*}R_{tqrs} + g_{qt}R_{purs} - g_{qu}R_{ptrs} \\ &+ g_{rt}R_{pqus} - g_{ru}R_{pqts} + g_{st}R_{pqru} - g_{su}R_{pqrt}) = 0\end{aligned} \tag{2.12}$$

Since $\delta_v \neq 0$ for some v, it follows from (2.12) that $R^* = 0$. thus K_{n-r} have zero scalar curvature. Equation (2.12) also yields:

$$g_{pt}R_{uqrs} - g_{p*}R_{tqrs} + g_{qt}R_{purs} - g_{qu}R_{ptrs} + g_{rt}R_{pqus} - g_{ru}R_{pqts} + g_{st}R_{pqru} - g_{su}R_{pqrt} = 0 \tag{2.13}$$

Multiplying by $g^{qr}g^{pt}$ (2.13) is contracted to

$$R_{us} = \frac{R}{r} g_{us}. \tag{2.14}$$

Also, multiplying (2.13) by g^{pt} and using (2.14), we have obtained:

$$rR_{uqrs} - 2R_{uqrs} + R_{rqus} + R_{sqru} = \frac{R}{r}(g_{su}g_{qr} - g_{ur}g_{qs}).$$

This gives:

$$rR_{uqrs} - R_{uqrs} + R_{uqsr} + R_{usrq} + R_{urqs} = \frac{R}{r}(g_{us}g_{qr} - g_{ur}g_{qs}).$$

That is,

$$R_{uqrs} = \frac{R}{r(r-1)}(g_{us}g_{qr} - g_{ur}g_{qs}).$$

We denote that the scalar curvature K_r as appears in (2.14), is constant. Hence we are lead to state that K_r has constant curvature.

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